A NOTE ON IMPORTANCE SAMPLING SIMULATION FOR A GERM-GRAIN MODEL

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Abstract. In this paper we demonstrate how to use the importance sampling method to simulate rare events in a germ-grain model. We analyze conditions under which two germ-grain models are mutually absolutely continuous. We also find the likelihood set process. We apply these results in simulating the probability that the radius of the occupied component of the origin in continuous percolation is greater than some R. This method is based on the reduction of the variance of estimator.

1. INTRODUCTION

A germ-grain model may provide a good description for a very irregular pattern observed in microscopy materials science, biology and analysis of images. Perhaps the best known model is the Boolean model (Matheron [5]) formalizing a configuration of independent, randomly placed particles. A Boolean model is formed by placing random balls centered at the points of a Poisson process and taking the union of these balls. The points of the Poisson process are sometimes called the *germs* and the associated balls the *grains*. In a natural generalization of the Boolean model the Poisson process of germs is replaced by a general point process and balls by any compact sets or even more general objects. If we take these objects as a mark at the point of the point process of germs, then such a marked point process N will be called a *marked point process* (abbreviated as m.p.p.) *driving the germ-grain model.*

The simulation of a Boolean model within a compact set $T \subset \mathbf{R}^d$ falls into the following stages. First, the number of points is determined by simulating a Poisson random variable J with parameter $\lambda |T|$, where λ is the intensity of

the Poisson process and $|T|$ the volume of the set T. Then J independent random points $\{t_i\}$ are simulated in T according to the Bernoulli process. Next, we generate J i.i.d. copies of radius m. Finally, the Boolean model is constructed by

$$\mathscr{A}(N) = \bigcup_{t_i \in T} (t_i \oplus m_i \mathcal{O}),$$

where $\oplus$ is the Minkowski addition, $\mathcal{O}$ is the unit ball, m_i an i.i.d. positive random variable, and $\{[t_i, m_i \mathcal{O}]\}$ is a realization of N. Denote by $P_N(\cdot)$ the distribution of $\mathscr{A}(N)$. That is, $\mathscr{A}(N)$ is a random element on $(\mathscr{F}, \mathscr{T}_f(R^d))$, where $\mathscr{F}$ is a family of closed sets and $\mathscr{T}_f(R^d)$ is the Fell σ-algebra generated by $\mathscr{F}_K = \{F \in \mathscr{F}: F \cap K \neq \varnothing\}$ for K ranged over all compact sets (see Matheron [5], Section 1-2).

We want to simulate the so-called rare event $A \in \mathscr{T}_f(R^d)$ for a Boolean model $\mathscr{A}(N)$ or, more generally, for a germ-grain model. That is, $P_N(A)$ is "small" (typically of order 10^{-6}). Using the so-called Crude Monte Carlo (CMC) method of simulation in this case is inefficient. Precisely, let n be the size of a sample and $1(A_1), 1(A_2), \ldots, 1(A_n)$ replicas of $1(A)$. Then estimating $p = P_N(A)$ by

$$\hat{p} = \frac{1}{n} \sum_{i=1}^{n} 1(A_i),$$

we make the relative error high:

$$\mathrm{Re}(p) := \frac{\sqrt{\hat{p}(1-\hat{p})}}{\hat{p}} \sim \frac{1}{\sqrt{\hat{p}}} \to +\infty \quad \text{as } \hat{p} \to 0.$$

Therefore we will use the Importance Sampling (IS) method. The main idea is to compute $P_N(A)$ by simulating a germ-grain model from a probability measure $P_{\tilde{N}}$ such that P_N is absolutely continuous with respect to it. In Proposition 2.1 and Theorem 2.1 we find sufficient conditions under which P_N is absolutely continuous with respect to $P_{\tilde{N}}$. We also find the Radon–Nikodym derivative $L(\cdot)$, that is

$$\frac{dP_N}{dP_{\tilde{N}}}(\Xi) = L(\Xi)$$

for realization $\Xi \in \mathscr{F}$ of $\mathscr{A}(N)$. Then

(1.1) $$P_N(A) = \int_A L(\Xi) \, dP_{\tilde{N}}(\Xi).$$

Hence to estimate $P_N(A)$ we generate n replicas $(1(A_1), L_1), (1(A_2), L_2), \ldots, (1(A_n), L_n)$ from the measure $P_{\tilde{N}}$ and construct the estimator

(1.2) $$\hat{p}_{IS} = \frac{1}{n} \sum_{i=1}^{n} L_i \, 1(A_i).$$

The 95% confidence interval is then

(1.3)
$$\hat{p}_{IS} \pm \frac{1.96}{\sqrt{n}} \hat{\sigma}_{IS},$$

where

(1.4)
$$\hat{\sigma}_{IS}^2 = \frac{1}{n} \sum_{i=1}^{n} L_i^2 \mathbf{1}(A_i) - \hat{p}_{IS}^2.$$

We choose the measure $P_{\tilde{N}}$ (that is, the parameters of the new germ-grain model) in such a way that the event A is observed frequently. In other words, under a good choice of the parameters we decrease the relative error.

We analyze in detail the Boolean model, where the point process constitutes a Poisson process and balls have radius $m = 1$. We consider the event A_0 such that balls form a chain (all circles in this chain are connected), which joins the origin with the border of box T with side length R. That is, we simulate the probability that the radius of the occupied component of the origin is greater than R. The problem of finding $P_N(A_0)$ is relevant in industry when we apply the electrodes to the plates of the dielectric materials. Because of the manufacturing process small holes arise in the electrodes. A chain of small holes crossing from the origin to the border of the box means a diminished value of the capacitance. Typically, the parameters of the model: size of T and intensity of the Poisson process λ are such that $P_N(A_0)$ is "small". We prove that the IS scheme for an appropriate choice of a new intensity of the Poisson process is logarithmically efficient, which implies improvement. We also give some numerical results.

The paper is organized as follows. In Section 2 we find sufficient conditions under which the two germ-grain models are absolutely continuous. In Section 3 we analyze in detail the example mentioned above.

2. GERM-GRAIN MODEL

We start with a formal definition of a general germ-grain model. We will define a marked point process as a point process on a product space of locations and marks with the additional property that the marginal location process is itself a well-defined point process. By $\mathscr{B}(X)$ we denote the Borel σ-field of X. The location space $(T, \mathscr{B}(T))$ is the compact subspace of the Polish space $(W, \mathscr{B}(W))$. The marks space $(\mathscr{K}, \mathscr{T}_k)$ is the space of all compact subsets of W with σ-algebra $\mathscr{T}_k := \mathscr{T}_f(W) \cap \mathscr{K}$, where $\mathscr{T}_f(W)$ is the Fell σ-algebra on W. Let $(M, \mathscr{M})$ be a measurable space of simple integer-valued measures which are finite on bounded sets. We conceive a marked point process N as a random

element on $(M, \mathcal{M}, P)$, that is

(2.1) $$N(\omega, B, C) = \sum_{n \geq 1} \varepsilon_{(t_n(\omega), m_n(\omega))}(B, C),$$

where $B \in \mathcal{B}(T)$, $C \in \mathcal{T}_k$, and $\varepsilon_{t_n, m_n}(\cdot, \cdot)$ is a Dirac measure. Note that $N^{\mathcal{K}}(\cdot) := N(\cdot, \mathcal{K})$ constitutes a simple point process on T.

Let $\mu(dt, dm)$ be a mean measure of N:

$$EN(B, C) = \mu(B, C), \quad B \in \mathcal{B}(T), \ C \in \mathcal{T}_k.$$

Similarly, let $\lambda(dt)$ be a mean measure of a point process $N^{\mathcal{K}}(\cdot)$:

$$EN^{\mathcal{K}}(B) = \lambda(B), \quad B \in \mathcal{B}(T).$$

It can be shown that $\mu(dt, dm)$ is absolutely continuous with respect to $\lambda(dt)$, that is by the Radon–Nikodym theorem there exists a density $v_t(dm)$ such that

(2.2) $$\mu(dt, dm) = \lambda(dt) v_t(dm),$$

where $v_t(dm)$ can be interpreted as the distribution of the mark of the point t.

EXAMPLE 2.1. If $N(\cdot, \cdot)$ is a marked Poisson process with mean measure $\lambda(\cdot)$ and i.i.d. marking, then $\mu(dt, dm) = \lambda(dt) v(dm)$, where $v(\cdot)$ is a distribution of mark. If $N^{\mathcal{K}}(\cdot)$ is a Poisson process on $\mathbf{R}^d$ with intensity λ, then $\mu(dt, dm) = \lambda \, dt \, v(dm)$.

Remark 2.1. In the classical theory of marked processes on the real line, it is well known that under certain conditions on the probability space and filtration, the mean measure of a marked point process determines its distribution (Jacod [3]). As we shall see, this is not true for processes on general spaces. Consider the Poisson process $N^{\mathcal{K}}(\cdot) = N(\cdot, \mathcal{K})$ on $T = [0, 1]^2$ with mean measure equal to Lebesgue measure $\lambda(dt) = dt$. However, the Lebesgue measure is also the mean measure of the following process. Let $\Phi_0, \Phi_1, \ldots$ be i.i.d. unit rate Poisson processes on $[0, 1]$. Denote by T_k^i the time of the kth jump of Φ_i. Now, let locations of the points be $\{(T_i^0, T_k^i), i, k \geq 1\}$.

Now, writing $B \oplus C = \{b + m : b \in B, m \in C\}$ for the Minkowski addition of B and C, we define a germ-grain model by the union:

$$\mathcal{A}(N) = \bigcup_{n \geq 1} (t_n \oplus m_n).$$

Consider two marked point processes N and $\tilde{N}$ on $T \times \mathcal{K}$ having the mean measures μ and $\tilde{\mu}$, respectively. Let P_N and $P_{\tilde{N}}$ be the distribution of the germ-grain model driven by m.p.p.'s N and $\tilde{N}$, respectively. Let us put

$$\mathcal{N} = \{n : P(N(T \times \mathcal{K}) = n) > 0\} \quad \text{and} \quad \tilde{\mathcal{N}} = \{n : P(\tilde{N}(T \times \mathcal{K}) = n) > 0\}.$$

For $n \in \mathcal{N}$ we define a conditional distribution $N_{|n}$ of N, given $N(T \times \mathcal{K}) = n$, that is,

$$N_{|n}(B, C) := E[N(B, C) \mid N(T, \mathcal{K}) = n], \qquad B \in \mathcal{B}(T), \ C \in \mathcal{T}_k.$$

Note that $N_{|n}$ is also a marked point process on $T \times \mathcal{K}$. Let $\mu_{|n}(dt, dm)$ be its mean measure. Similarly we define $\tilde{N}_{|n}(\cdot, \cdot)$ and $\tilde{\mu}_{|n}(\cdot, \cdot)$.

PROPOSITION 2.1. *The marked point process $N(\cdot, \cdot)$ is absolutely continuous with respect to the marked point process $\tilde{N}(\cdot, \cdot)$ ($N \ll \tilde{N}$) iff $\mu_{|n} \ll \tilde{\mu}_{|n}$ for $n \in \mathcal{N}$ and $N(T \times \mathcal{K}) \ll \tilde{N}(T \times \mathcal{K})$.*

Proof. If $N \ll \tilde{N}$, then $N(T \times \mathcal{K}) \ll \tilde{N}(T \times \mathcal{K})$ and $N_{|n} \ll \tilde{N}_{|n}$, and hence also $\mu_{|n} \ll \tilde{\mu}_{|n}$. We prove the converse implication. We use the notation

$$(\boldsymbol{t}, \boldsymbol{m}) = (t_1, m_1, \ldots, t_k, m_k)$$

for the $(T \times \mathcal{K})^k$-valued vectors ($k = 1, 2, \ldots$). The kth order factorial measure $\alpha_{|n}^k$ of $N_{|n}$ is a measure on $(T \times \mathcal{K})^k$ defined by

$$\alpha_{|n}^k\big(d(\boldsymbol{t}, \boldsymbol{m})\big) = E N_{|n}^k\big(d(\boldsymbol{t}, \boldsymbol{m})\big),$$

where

$$N_{|n}^k\big(d(\boldsymbol{t}, \boldsymbol{m})\big) = N_{|n}\big(d(t_1, m_1)\big)(N_{|n} - \varepsilon_{(t_1, m_1)})\big(d(t_2, m_2)\big) \ldots \Big(N_{|n} \sum_{i=1}^{k-1} \varepsilon_{(t_i, m_i)}\Big)\big(d(t_k, m_k)\big).$$

We prove that $\alpha_{|n}^k$ is absolutely continuous with respect to $\tilde{\alpha}_{|n}^k$ for each $n \in \mathcal{N} \cap \tilde{\mathcal{N}}$ and $k \leqslant n$. We use induction. For $k = 1$ the assertion is satisfied, since $\alpha_{|n}^1(dt, dm) = \mu_{|n}(dt, dm)$ and $\tilde{\alpha}_{|n}^1(dt, dm) = \tilde{\mu}_{|n}(dt, dm)$. Assume that the assertion is satisfied for $k-1$. Let

$$(2.3) \qquad\qquad L_{k-1}(\boldsymbol{t}, \boldsymbol{m}) = \frac{d\alpha_{|n}^{k-1}}{d\tilde{\alpha}_{|n}^{k-1}}(\boldsymbol{t}, \boldsymbol{m}).$$

Then

$$(2.4) \qquad\qquad L_1(\boldsymbol{t}, \boldsymbol{m}) = \frac{d\mu_{|n}}{d\tilde{\mu}_{|n}}(\boldsymbol{t}, \boldsymbol{m}).$$

Denote by $\sum^{\#}$ the summation over distinct points in T. Then

$$\alpha_{|n}^k(B_1 \times C_1, \ldots, B_k \times C_k)$$

$$= \int_{T \times \mathcal{K}} \int_{(T \times \mathcal{K})^{k-1}} \sum_{t_1, \ldots, t_{k-1}}^{\#} \sum_{t_k \neq t_i, i \leqslant k-1} \varepsilon_{(t_1, m_1)}(B_1 \times C_1) \ldots \varepsilon_{(t_k, m_k)}(B_k \times C_k)$$

$$\times d\alpha_{|n}^{k-1}(t_1, m_1, \ldots, t_{k-1}, m_{k-1}) \, d\mu_{|n}(t_k, m_k)$$

$$= \int_{T \times \mathcal{K}} \int_{(T \times \mathcal{K})^{k-1}} \sum_{t_1, \ldots, t_{k-1}}^{\#} \sum_{t_k \neq t_i, i \leqslant k-1} L_{k-1}(t_1, m_1, \ldots, t_{k-1}, m_{k-1}) L_1(t_k, m_k)$$

$$\times \varepsilon_{(t_1, m_1)}(B_1 \times C_1) \ldots \varepsilon_{(t_k, m_k)}(B_k \times C_k)$$

$$\times d\tilde{\alpha}_{|n}^{k-1}(t_1, m_1, \ldots, t_{k-1}, m_{k-1}) d\tilde{\mu}_{|n}(t_k, m_k)$$

$$= \int_{(T \times \mathcal{K})^k} \sum_{t_1, \ldots, t_k}^{\#} L_k(t, m) \varepsilon_{(t_1, m_1)}(B_1 \times C_1) \ldots \varepsilon_{(t_k, m_k)}(B_k \times C_k) \tilde{\alpha}_{|n}^k(d(t, m)),$$

where $L_k(t, m) = L_{k-1}(t_1, m_1, \ldots, t_{k-1}, m_{k-1}) L_1(t_k, m_k)$. Thus $\alpha_{|n}^k \ll \tilde{\alpha}_{|n}^k$ for $k \leqslant n$. Since $N(T \times \mathcal{K}) \ll \tilde{N}(T \times \mathcal{K})$, there exists nonnegative D_n such that

$$P\big(N(T \times \mathcal{K}) = n\big) = D_n P\big(\tilde{N}(T \times \mathcal{K}) = n\big).$$

Denote by $Q_{|n}$ the distribution of $N_{|n}$ on $(M, \mathcal{M})$. For a bounded real-valued $\mathcal{M}$-measurable function f we have

$$Ef(N) = \sum_{n \in \mathcal{N}} P\big(N(T \times \mathcal{K}) = n\big) \int_M f(t, m) Q_{|n}\big(d(t, m)\big)$$

$$= \sum_{n \in \mathcal{N}} P\big(N(T \times \mathcal{K}) = n\big) \int f(t, m) \alpha_{|n}^n\big(d(t, m)\big)$$

$$= \sum_{n \in \tilde{\mathcal{N}}} P\big(\tilde{N}(T \times \mathcal{K}) = n\big) \int f(t, m) D_n L_n(t, m) \tilde{\alpha}_{|n}^n\big(d(t, m)\big) = Ef(\tilde{N}) L(\tilde{N}),$$

where

$$L\Big(\sum_{i=1}^n \varepsilon_{(t_i, m_i)}(\cdot, \cdot)\Big) = D_n L_n(t, m)$$

for realization $\sum_{i=1}^n \varepsilon_{(t_i, m_i)}(\cdot, \cdot)$ of the marked point process $\tilde{N}$, and $f(t, m)$ means $f\big(\sum_{i=1}^k \varepsilon_{(t_i, m_i)}(\cdot, \cdot)\big)$. This completes the proof. ■

Remark 2.2. This result is well known for Poisson processes; see Matthes et al. [6], Proposition 1.7.11.

Remark 2.3. For a marked Poisson process with intensity λ and a mark independent of a position with a distribution measure $\nu(\cdot)$, we have

$$P\big(N(T \times \mathcal{K}) = n\big) = \frac{(\lambda |T|)^n}{n!} e^{-\lambda |T|},$$

where $|B|$ is the volume of a set B, and

$$\alpha_{|n}^n\big(d(t, m)\big) = dt_1 \ldots dt_n d\nu(m_1) \ldots d\nu(m_n).$$

Remark 2.4. Note that from the assumptions of Proposition 2.1 it follows that μ is absolutely continuous with respect to $\tilde{\mu}$. In fact, let $B \times C$ be such that $\tilde{\mu}(B \times C) = 0$. We have $\mathcal{N} \subseteq \tilde{\mathcal{N}}$. Then

$$\sum_{n \in \tilde{\mathcal{N}}} P\big(\tilde{N}(T \times \mathcal{K}) = n\big) \tilde{\mu}_{|n}(B \times C) = 0$$

and for $n \in \tilde{\mathcal{N}}$ we have $\tilde{\mu}_{|n}(B \times C) = 0$. Hence also $\mu_{|n}(B \times C) = 0$ for $n \in \mathcal{N}$, and finally $\mu(B \times C) = 0$. The converse statement in general is not true. In fact, consider two point processes $N^{\mathcal{X}}(\cdot)$ and $\tilde{N}^{\mathcal{X}}(\cdot)$ on $T = [0, 1]^2$ given by

$$N^{\mathcal{X}}(\cdot) = \varepsilon_{(1/2, 1/2)}(\cdot) \quad \text{and} \quad \tilde{N}^{\mathcal{X}}(\cdot) = \varepsilon_{(1/2, 1/2)}(\cdot) + \varepsilon_{(1/4, 1/4)}(\cdot).$$

THEOREM 2.1. *If* $N \ll \tilde{N}$, *then* $P_N \ll P_{\tilde{N}}$.

Proof. Suppose that $P_{\tilde{N}}(\Lambda) = 0$ for $\Lambda \in \mathscr{T}_f(W)$. That is,

$$\sum_{n \in \tilde{\mathcal{N}}} P\big(\tilde{N}(T \times \mathcal{X}) = n\big) \int \mathbf{1}_\Lambda \Big(\bigcup_{i=1}^n (t_i \oplus m_i) \Big) \tilde{N}_{|n}^n \big(d(t, m)\big)$$

$$= \sum_{n \in \tilde{\mathcal{N}}} P\big(\tilde{N}(T \times \mathcal{X}) = n\big) \int_{(T \times \mathcal{X})^n} \mathbf{1}_\Lambda \Big(\bigcup_{i=1}^n (t_i \oplus m_i) \Big) \tilde{\alpha}_{|n}^n \big(d(t, m)\big) = 0,$$

where $\mathbf{1}_\Lambda(F) = 1$ if $F \in \Lambda$ and $\mathbf{1}_\Lambda(F) = 0$ otherwise. Hence all terms must be zero and for all $n \in \mathcal{N}$ we have

$$(2.5) \qquad \int_{(T \times \mathcal{X})^n} \mathbf{1}_\Lambda \Big(\bigcup_{i=1}^n (t_i \oplus m_i) \Big) \tilde{\alpha}_{|n}^n \big(d(t, m)\big) = 0.$$

Thus from Proposition 2.1 for $n \in \mathcal{N}$ we obtain

$$(2.6) \qquad \int_{(T \times \mathcal{X})^n} \mathbf{1}_\Lambda \Big(\bigcup_{i=1}^n (t_i \oplus m_i) \Big) \alpha_{|n}^n \big(d(t, m)\big) = 0$$

yielding $P_N(\Lambda) = 0$. This completes the proof. ∎

The likelihood ratio $dP_{\tilde{N}}/dP_N(\Xi)$ for realization $\Xi \in \mathscr{F}$ of $\mathscr{A}(N)$ is

$$(2.7) \quad L(\Xi) := \frac{dP_N}{dP_{\tilde{N}}}(\Xi)$$

$$= \frac{\sum_{n \in \mathcal{N}} P\big(N(T \times \mathcal{X}) = n\big) \int_{(T \times \mathcal{X})^n} I_\Xi \big(\bigcup_{i=1}^n (t_i \oplus m_i) \big) \alpha_{|n}^n \big(d(t, m)\big)}{\sum_{n \in \tilde{\mathcal{N}}} P\big(\tilde{N}(T \times \mathcal{X}) = n\big) \int_{(T \times \mathcal{X})^n} I_\Xi \big(\bigcup_{i=1}^n (t_i \oplus m_i) \big) \tilde{\alpha}_{|n}^n \big(d(t, m)\big)},$$

where $I_\Xi(F) = 1$ if $F = \Xi$ and $I_\Xi = 0$ otherwise.

EXAMPLE 2.2 (Poisson cluster process). Let $N(\cdot, \cdot)$ be a Poisson process on a compact set $T \subset \mathbf{R}^d$ with intensity λ marked by a point process $N_i(\cdot)$ on a compact set $E \subset \mathbf{R}^d$ at position t_i, where $N_i(\cdot)$ are conditionally independent, given the realization of the parent Poisson process. Then $\bar{N}(\cdot) = \bigcup_{i=1}^\infty (t_i \oplus N_i(\cdot))$ is a cluster Poisson process (see [4]). Assume that $N_i(\cdot)$ are absolutely continuous with respect to a unit rate Poisson process on E with the offspring density $\nu(\cdot)$. Let $\tilde{N}(\cdot)$ be the Poisson cluster process with intensity $\tilde{\lambda}$ and the offspring density $\tilde{\nu}(\cdot)$. Conditioning on parent point configuration t, the

offspring $\bigcup_{i=1}^{\infty} (t_i \oplus N_i))$ is absolutely continuous to a unit rate Poisson process on $T \oplus E$ with conditional density

$$e^{|T \oplus E|} \sum_{\phi} \prod_{i=1}^{n} \left[\nu \left(\Xi_{\phi^{-1}(i)} - t_i \right) e^{-|E|} \right],$$

where the sum is over all ordered n partitions ϕ of Ξ. Hence taking expectation over t gives

(2.8) $\quad \dfrac{d\bar{N}}{d\tilde{\bar{N}}}(\Xi) = L(\Xi)$

$$= \frac{\sum_{n=1}^{\infty} (\lambda^n/n!)\, e^{-\lambda|T|}\, e^{|E|(1-n)} \sum_{\phi} \int_{T^n} \prod_{i=1}^{n} \nu\left(\Xi_{\phi^{-1}(i)} - t_i\right) dt_1 \dots dt_n}{\sum_{n=1}^{\infty} (\tilde{\lambda}^n/n!)\, e^{-\tilde{\lambda}|T|}\, e^{|E|(1-n)} \sum_{\phi} \int_{T^n} \prod_{i=1}^{n} \tilde{\nu}\left(\Xi_{\phi^{-1}(i)} - t_i\right) dt_1 \dots dt_n};$$

see also Van Lieshout [8]. If each parent point has a single daughter point with displacement densities $\nu(\cdot)$ and $\tilde{\nu}(\cdot)$, then (2.8) reduces to

$$L(\Xi) = e^{(\tilde{\lambda} - \lambda)|T|} \left(\frac{\lambda}{\tilde{\lambda}} \right)^{H(\Xi)} \prod_{i=1}^{H(\Xi)} \frac{\int_T \nu(\Xi_i - t)\, dt}{\int_T \tilde{\nu}(\Xi_i - t)\, dt},$$

where $H(\Xi)$ denotes the number of points in configuration Ξ, and Ξ_i is the ith point of Ξ.

EXAMPLE 2.3 (wire frame model). The germ-grain model in which there is a one-to-one correspondence between the driving m.p.p. and the germ-grain model itself will be called a *wire frame model*. The classical example is a Boolean sphere model on $\mathbf{R}^d$, in which marks are spheres centered at the location points $\{t_i\}$ of the point process $N^{\mathscr{K}}(\cdot)$. Denote by $\pi\colon T \oplus \mathscr{K} \to T \times \mathscr{K}$ the one-to-one mapping such that $\pi(\mathscr{A}(N)) = N$. Similarly we define the mapping $\pi^{\mathscr{K}}(\mathscr{A}(N)) = N^{\mathscr{K}}$. Then from (2.7) we have

(2.9) $$\frac{dP_N}{dP_{\tilde{N}}}(\Xi) = L(\Xi) = \frac{dN}{d\tilde{N}}(\pi(\Xi)).$$

Consider the marked Poisson processes $N(\cdot,\cdot)$ and $\tilde{N}(\cdot,\cdot)$ with mean measures $\lambda(\cdot)$ and $\tilde{\lambda}(\cdot)$, respectively, and with independent marking with measures $\nu(\cdot)$ and $\tilde{\nu}(\cdot)$. Let $\lambda \ll \tilde{\lambda}$ and $\nu \ll \tilde{\nu}$. Then, by Proposition 6.10 of Karr [4], p. 232, we have $N \ll \tilde{N}$, and hence also $P_N \ll P_{\tilde{N}}$. In this case we have

(2.10) $\quad \dfrac{dP_N}{dP_{\tilde{N}}}(\Xi)$

$$= \exp\left\{ \int \log\left(\frac{d\lambda}{d\tilde{\lambda}}(t) \right) d\pi^{\mathscr{K}}(\Xi)(t) \right\} \exp\left\{ \int_T \left(1 - \frac{d\lambda}{d\tilde{\lambda}}(t) \right) d\tilde{\lambda}(t) \right\} \prod_{i=1}^{H(\Xi)} \frac{d\nu}{d\tilde{\nu}}(t_i),$$

where $H(\Xi)$ is the number of points of the Poisson process constructing a set Ξ. In particular, if $N(\cdot,\cdot)$ and $\tilde{N}(\cdot,\cdot)$ are marked Poisson processes with inten-

sities λ and $\tilde{\lambda}$, respectively, and the diameters in both Boolean sphere models have the same distribution, then

$$(2.11) \qquad \frac{dP_N}{dP_{\tilde{N}}}(\Xi) = \exp\{(\tilde{\lambda}-\lambda)|T|\}\left(\frac{\lambda}{\tilde{\lambda}}\right)^{H(\Xi)}.$$

3. RADIUS OF THE OCCUPIED COMPONENT OF THE ORIGIN

In this section we consider the Boolean sphere model on R^d driven by the marked Poisson process $N(\cdot,\cdot)$ with intensity λ and marks being i.i.d. spheres. A marked Poisson process $\tilde{N}(\cdot,\cdot)$ has the intensity $\tilde{\lambda}$ and marks being spheres distributed like in a Boolean sphere model governed by P_N. We will consider a rare event $A \in \mathcal{T}_f(R^d)$ for which $\lim_{\lambda\to 0} P_N(A) = 0$. By (2.11) we have

$$L(\Xi) = \exp\{(\tilde{\lambda}-\lambda)|T|\}(\lambda/\tilde{\lambda})^{H(\Xi)},$$

where $H(\Xi)$ is a number of points of Poisson process in T when realization of $\mathcal{A}(N)$ is Ξ. We will give now an example of A for which the IS scheme works well, that is, it reduces the relative error.

DEFINITION 3.1. We say that the IS scheme is *logarithmically efficient* if

$$\liminf_{\lambda\to 0}\frac{\log \operatorname{Var}\hat{p}_{\mathrm{IS}}}{\log p^2} \geq 1.$$

If

$$\limsup_{\lambda\to 0}\frac{\operatorname{Var}\hat{p}_{\mathrm{IS}}}{p} = 0,$$

then the IS scheme is an *improvement over CMC simulation.*

Logarithmic efficiency implies improvement (see Asmussen [1]).

THEOREM 3.1. *Let A be an event such that*

$$(3.1) \qquad \lim_{\lambda\to 0}\left(-\frac{1}{\beta(\lambda)}\right)\log P_\lambda(A) = 1$$

for some positive scaling function $\beta(\cdot)$. Let $\tilde{\lambda} > \lambda$. If

$$(3.2) \qquad \lim_{\lambda\to 0}\frac{\beta(\lambda^2/\tilde{\lambda})}{2\beta(\lambda)} \geq 1,$$

then the IS scheme is logarithmically efficient. If $\beta(\cdot)$ is strictly decreasing so that

$$(3.3) \qquad \lim_{\lambda\to 0}[\beta(\lambda^2/\tilde{\lambda})-\beta(\lambda)] = \infty,$$

then the IS scheme is an improvement.

Proof. Let $\bar{\lambda} = \lambda^2/\tilde{\lambda}$. Denote by P_β the distribution of $\mathscr{A}(N)$ when $N^{\mathscr{K}}(\cdot)$ is the Poisson process with intensity β. For $A \in \mathscr{T}_f(R^d)$ we have

$$E_{\tilde{\lambda}}[L^2; A] = \exp\{(\tilde{\lambda}-\lambda)|T|\} \int_A \left(\frac{\lambda}{\tilde{\lambda}}\right)^{H(\Xi)} dP_\lambda(\Xi)$$

$$= \exp\{(\tilde{\lambda}-\lambda)|T|\} \exp\{(\bar{\lambda}-\lambda)|T|\} \int_A \left(\frac{\lambda}{\bar{\lambda}}\right)^{H(\Xi)} \left(\frac{\lambda}{\tilde{\lambda}}\right)^{H(\Xi)} dP_{\bar{\lambda}}(\Xi)$$

$$= \exp\left\{\frac{1}{\tilde{\lambda}}(\tilde{\lambda}-\lambda)^2 |T|\right\} P_{\bar{\lambda}}(A).$$

Thus

$$\lim_{\lambda \to 0} \frac{\log \mathbf{Var}\, \hat{p}_{IS}}{\log p^2} = \lim_{\lambda \to 0} \frac{\log E_{\tilde{\lambda}}[L^2; A]}{-2\beta(\lambda)}$$

$$= \lim_{\lambda \to 0} \frac{\tilde{\lambda}|T| + \log P_{\bar{\lambda}}(A)}{-2\beta(\lambda)} = \lim_{\lambda \to 0} \frac{\tilde{\lambda}|T| - \beta(\bar{\lambda})}{-2\beta(\lambda)} = \lim_{\lambda \to 0} \frac{\beta(\bar{\lambda})}{2\beta(\lambda)} \geqslant 1.$$

Similarly we prove that under (3.3) the IS scheme is an improvement. ∎

Assume from now that the radius of spheres is equal to $m = 1$. Let $T = T(R) := [-R, R]^d \subseteq R^d$ ($d \geqslant 2$) for fixed $R > 1$. We shall apply Theorem 3.1 to the event $A_0 := \{0 \leftrightarrow \partial T(R)\}$ that there exists a path through balls of the Boolean model joining 0 with the surface ∂T of T:

$$\partial T(R) = \{t = (t_1, \ldots, t_d) \in R^d : \max_i |t_i| = R\}.$$

In other words, in the wire frame model, the origin is inside a sphere which is connected through a chain of spheres with the surface ∂T.

THEOREM 3.2. *The IS scheme for A_0 is logarithmically efficient.*

For the proof we need the following lemmas.

LEMMA 3.1. *There exists a decreasing positive function $\phi(\lambda)$ such that*

$$(3.4) \qquad \lim_{\lambda \to 0} \left(-\frac{1}{R\phi(\lambda)}\right) \log P_\lambda(0 \leftrightarrow \partial T(R)) = 1.$$

Proof. Using the version BK and FKG inequalities for continuous percolation (see Theorems 2.2 and 2.3 of Meester and Roy [7]) we obtain (6.24) in Grimmett [2]. Then using the subadditive inequality limit theorem one can mimic the proof of Theorem 6.10 of Grimmett [2] to obtain its version for the continuous percolation. That is, there exist strictly positive constants ϱ and σ, independent of λ, and a decreasing positive function $\phi(\lambda)$ such that

$$\varrho R^{1-d} e^{-R\phi(\lambda)} \leqslant P_\lambda(0 \leftrightarrow \partial T(R)) \leqslant \sigma R^{d-1} e^{-R\phi(\lambda)} \quad \text{for all } R > 1.$$

This completes the proof. ∎

LEMMA 3.2. *Let* $0 \leqslant f(x) \leqslant 1$. *Then*

$$(3.5) \qquad \lim_{\lambda \to 0} \frac{\phi(\lambda f(\lambda)) \log(\lambda)}{\phi(\lambda) \log(\lambda f(\lambda))} \geqslant 1.$$

Proof. The main idea of the proof is to approximate the continuous problem by site percolation problems on a special lattice, constructed by partitioning R^d into small cubes. Let κ be a positive integer and $Z_\kappa^d = \kappa^{-1} Z^d$. We partition R^d into cubes whose centers are the points of Z_κ^d, defining

$$B_\kappa(t) = \mathop{\times}_{i=1}^{d} \left[t_i - \frac{1}{2\kappa}, \, t_i + \frac{1}{2\kappa} \right] \quad \text{for } t \in Z_\kappa^d.$$

We turn Z_κ^d into a lattice $\mathscr{G}_\kappa$ by defining the adjacency relation $\sim$ on Z_κ^d with the rule that $x \sim y$ iff there exist points $u \in B_\kappa(x)$ and $v \in B_\kappa(y)$ such that $\varrho(u, v) \leqslant 2$, where $\varrho(\cdot, \cdot)$ is the Euclidean distance. We shall consider site percolation on the ensuing lattice $\mathscr{G}_\kappa$. We declare a vertex x of $\mathscr{G}_\kappa$ to be open if there exist one or more points of the Poisson process within the cube $B_\kappa(x)$, and closed otherwise. The states of different vertices are independent random variables and the probability $p_\kappa(\lambda)$ that any given vertex is open is given by

$$(3.6) \qquad p_\kappa(\lambda) = 1 - \exp(-\lambda \kappa^{-d}).$$

Let $\gamma_\kappa = (1 + \kappa^{-1} \sqrt{d})^d$. From the rescaling property of a Boolean sphere model and the considerations made by Meester and Roy [7], p. 60, or Grimmett [2], Section 12.10, for sufficiently small λ we have

$$(3.7) \qquad P^s_{p_\kappa(\lambda \gamma_\kappa)}(0 \leftrightarrow \partial T(R)) \leqslant P_\lambda(0 \leftrightarrow \partial T(R)) \leqslant P^s_{p_\kappa(\lambda)}(0 \leftrightarrow \partial T(R)),$$

where $P^s_p(\cdot)$ is a law of site percolation on $\mathscr{G}_\kappa$ defined by the adjacency relation $\sim$, where the probability that a given vertex is open equals p. Thus, by Theorem 2.38 of Grimmett [2] applied to site percolation, we have

$$\frac{\log P_{\lambda f(\lambda)}(0 \leftrightarrow \partial T(R)) \log \lambda}{\log P_\lambda(0 \leftrightarrow \partial T(R)) \log \lambda f(\lambda)}$$
$$\geqslant \frac{\log P^s_{p_\kappa(\lambda f(\lambda))}(0 \leftrightarrow \partial T(R)) \log \lambda}{\log P^s_{p_\kappa(\lambda \gamma_\kappa)}(0 \leftrightarrow \partial T(R)) \log \lambda f(\lambda)} \geqslant \frac{\log p_\kappa(\lambda f(\lambda)) \log \lambda}{\log p_\kappa(\lambda \gamma_\kappa) \log \lambda f(\lambda)}.$$

Note that

$$\lim_{\alpha \to 0} \frac{\log p_\kappa(\alpha)}{\log \alpha} = 1.$$

This completes the proof. ∎

Proof of Theorem 3.2. Let $f(\lambda) = \lambda / \tilde{\lambda} < 1$. Then from Lemmas 3.1 and 3.2 we have

$$\lim_{\lambda \to 0} \frac{\log P_{\tilde{\lambda}}(0 \leftrightarrow \partial T(R))}{2 \log P_\lambda(0 \leftrightarrow \partial T(R))} = \lim_{\lambda \to 0} \frac{\phi(\lambda f(\lambda))}{2\phi(\lambda)} \geqslant \lim_{\lambda \to 0} \frac{\log(\lambda f(\lambda))}{2 \log \lambda} = 1,$$

which completes the proof of the theorem. ∎

Although the flavor of this section is not numerical we add for complete-
ness some numerical results. We made $10\,000$ simulations for $R = 64$, $d = 2$,
$m = 1$ and $\tilde{\lambda} = 1.7$. The percentage denotes the relative half-width of 95%
confidence interval based on the normal distribution.

Table 1. Simulation of the
event $0 \leftrightarrow \partial T(R)$

λ	p_{IS}
1.6	$7.5 \cdot 10^{-6} \pm 15.7\%$
1.5	$2.5 \cdot 10^{-6} \pm 32.1\%$
1.45	$1.2 \cdot 10^{-6} \pm 23.4\%$

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